RELATIVITY AND COSMOLOGY I

Problem Set 1 Fall 2023

1. The scales of the Universe - On the board

Planck's constant, the speed of light and Newton's gravitational constant

$$\hbar \approx 1.05 \times 10^{-34} \frac{\mathrm{m}^2\ \mathrm{kg}}{\mathrm{s}}\,, \qquad c \approx 3.00 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}}\,, \qquad G_N \approx 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg}\ \mathrm{s}^2}\,, \qquad (1)$$

can be used to construct quantities with units of length, time and mass.

(a) Find these quantities, called the Planck length, time and mass, and evaluate them in SI units. Do you notice a hierarchy?

In High Energy Physics (HEP), we use units such that $\hbar = c = 1$. These are called *natural units*. In this system, there is only one unit left to choose, and the standard choice is to use energy measured in Giga-electronVolts (GeV).

(b) What are 1m, 1s and 1kg equivalent to, in natural units?

In natural units, (almost) all physical quantities can be estimated in terms of:

- the mass of a nucleon[1]: $m_n \approx 1\,\mathrm{GeV}$,
- the mass of the electron: $m_e \approx 5 \times 10^{-4}\,\mathrm{GeV}$,
- the fine structure constant: $\alpha \approx \frac{1}{137}$,
- the Planck mass: $M_P = G_N^{-1/2} \approx 10^{19}\,\mathrm{GeV}$.

(c) Use these quantities to estimate the order of magnitude of:

- **Atoms**: estimate the size, mass and binding energy of atoms.
- **Solids**: estimate the density, typical pressure and speed of sound of a material made of atoms.
- **Planets**: estimate the size and mass of planets made of atoms.
- **Neutron stars**: estimate the size and mass of a neutron star.
- **Living beings**: estimate the maximal size and mass of beings living on the surface of a planet.

Hint: You can watch this inspiring lecture https://pirsa.org/10080006 by Nima Arkani-Hamed.

[1]For most estimates we can neglect the mass difference between protons and neutrons.

2. Index Gymnastics: Part 1

In this exercise we propose a set of quick questions that test your skills with indices.

(a) Identify the free and dummy indices in the following equations and change them into equivalent expressions with different indices. How many independent scalar equations does each expression represent in 4 spacetime dimensions?

- $A_\alpha B^\alpha = 5\,,$
- $A^\mu = \Lambda^\mu{}_\nu A^\nu\,,$
- $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\alpha\gamma}\,.$

(b) What is the value of δ^μ_μ and $\eta^\mu{}_\mu$in 4 dimensions? And in n dimensions?

In this course, we will use special notation to indicate the symmetrization or antisymmetrization of some indices in a tensor:

$$T_{(\mu_1\cdots\mu_n)} \equiv \frac{1}{n!}\sum_{\sigma\in P} T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\,, \qquad T_{[\mu_1\cdots\mu_n]} \equiv \frac{1}{n!}\sum_{\sigma\in P} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\,, \tag{2}$$

where P is the group of permutations, and the sign[2] of σ depends on whether the permutation is made of an even number of inversions (exchange of two elements) or not, with $\mathrm{sgn}(\sigma_{\text{even}}) = +1$ and $\mathrm{sgn}(\sigma_{\text{odd}}) = -1$. For example,

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)\,, \qquad T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)\,. \tag{3}$$

(c) Expand the following expressions

$$A_{\tau(\alpha} B^\tau{}_{\beta)}\,, \qquad A_{(\mu} B^\nu{}_{\tau\sigma)}\,, \qquad A_{[\mu\tau\sigma]}\,, \qquad A_{(\mu}{}^{[\nu} B^{\sigma]}{}_{\tau)}\,. \tag{4}$$

3. Canonical Form of Four-vectors

In special relativity, boosts, parametrized by the rapidity η through the relation

$$\frac{v}{c} = \tanh(\eta), \tag{5}$$

are realized by the following matrix

$$\Lambda = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) & 0 & 0 \\ -\sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

so that a four-vector which in the frame F has coordinates V^μ with $\mu = 0, 1, 2, 3$, will have coordinates $(\Lambda V)^\mu$ in a frame F' moving at speed v with respect to F.

[2]Further reading: https://en.wikipedia.org/wiki/Parity_of_a_permutation

(a) Show that for a time-like four-vector V^μ ($\eta_{\mu\nu}V^\mu V^\nu < 0$), one can always find a Lorentz transformation[3] Λ such that[4]

$$\Lambda V = \begin{pmatrix} \sqrt{-\eta_{\mu\nu}V^\mu V^\nu} \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{8}$$

(b) Show that for a space-like four-vector V^μ ($\eta_{\mu\nu}V^\mu V^\nu > 0$), one can always find a Lorentz transformation Λ such that

$$\Lambda V = \begin{pmatrix} 0 \\ \sqrt{\eta_{\mu\nu}V^\mu V^\nu} \\ 0 \\ 0 \end{pmatrix} . \tag{9}$$

(c) Show that for a light-like four-vector V^μ ($\eta_{\mu\nu}V^\mu V^\nu = 0$), one can always find a Lorentz transformation Λ such that

$$\Lambda V = \begin{pmatrix} V^0 \\ V^0 \\ 0 \\ 0 \end{pmatrix} . \tag{10}$$

These are the standard frames in which time-like, space-like and light-like four-vectors are usually expressed.

4. A Primer in Variational Calculus

A functional F is a map from the space of functions to the set of real (or complex) numbers. For example, in physics, the action is defined through a functional. Functional derivatives are defined as follows

$$\frac{\delta F[f(x)]}{\delta f(x_0)} = \lim_{\epsilon\to 0} \frac{F[f(x)+\epsilon\delta(x-x_0)] - F[f(x)]}{\epsilon} . \tag{11}$$

This is the functional derivative of the functional F with respect to the function $f(x)$ evaluated at point $x = x_0$. The x that appears in (11) is instead the variable of integration in the functional.

(a) Use this definition to compute the functional derivatives of the following functionals

$$\begin{aligned} F_1[f(x)] &= \int f(x)\mathrm{d}x \,, \qquad F_2[f(x)] = \int (f(x))^p \phi(x)\mathrm{d}x \,, \\ F_3[f(x)] &= \int g[f(x)]\mathrm{d}x \,, \quad F_4[x(t)] = \int \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \mathrm{d}t \,, \end{aligned} \tag{12}$$

[3] A Lorentz transformation is in general a combination of a rotation and a boost.

[4] **Hint:** you may want to use the following relations

$$\cosh(\tanh^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \,, \qquad \sinh(\tanh^{-1}(x)) = \frac{x}{\sqrt{1-x^2}} . \tag{7}$$

This concept can be generalized to tensors. For example,

$$\frac{\delta F[f^\nu(x)]}{\delta f^\mu(x_0)} = \lim_{\epsilon\to 0} \frac{F[f^\nu(x) + \epsilon\delta^\nu_\mu \delta(x - x_0)] - F[f^\nu(x)]}{\epsilon}\,, \tag{13}$$

where δ^ν_μ is effectively a Kronecker delta.

(b) Use this definition to compute the functional derivatives of the following functionals[5]

$$F_1[A^\nu] = \int A_\mu A^\mu \mathrm{d}x\,, \quad F_2[A^\nu] = \int F_{\mu\nu} F^{\mu\nu} d^4x\,. \tag{14}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

(c) Can you guess the definition of the functional derivative with respect to a two-index tensor $\frac{\delta F[g^{\mu\nu}(x)]}{\delta g^{\mu\nu}(x_0)}$?

[5]There are many conventions for writing the argument of a functional. Here, we omit the fact that $A^\nu(x)$ is a function of x. Sometimes, even indices are omitted.